An Extended Family of Measures for Directed Networks

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Abstract

Centrality indices such as β-centrality, Katz status, and Hubbell's index are commonly generalized to directed networks by relating the in-centrality of nodes to the in-centrality of their in-neighbors and equivalently so for out-centrality. This paper proposes an extension of Bonacich's β-centrality and related measures for directed networks where the in-centrality of a node depends on the out-centrality of their in-neighbors and their out-centrality on the in-centrality of their out-neighbors. The so defined indices extend hubs and authorities in the same way as β-centrality generalizes eigenvector centrality. Several technical results are presented including the extension of the range of permissible β parameters to negative values, similar to traditional β-centrality.

Keywords: centrality, directed graphs, singular value decomposition, exchange networks

1 Introduction

The traditional centrality measure by Katz (1953) and the related measure proposed by Hubbell (1965) both use a parameter β, to weight the influence of nodes at a greater distance. Bonacich (1987) took a similar approach with his β-centrality measure but suggested extending the range of β to include negative values so that being connected to a highly central actor had a negative impact. He gave the practical example of exchange networks to show when this might occur (Cook et al., 1983). In such settings, power comes from being connected to the powerless, and being connected to less powerful actors increases one's bargaining position.

The usual directed extensions of all three indices relate the in-centrality of nodes with the in-centrality of their in-neighbors and equivalently so for out-centrality. In this work, we extend the three measures to directed networks by assuming that the in-centrality of a node is dependent on the out-centrality of its in-neighbors and the out-centrality on the in-centrality of its out-neighbors. In doing so, we generalize Kleinberg's hubs and authorities (Benzi et al., 2013; Kleinberg et al., 1999) in the same way as β-centrality generalizes eigenvector centrality.

The remainder of the paper is organized as follows. We start by introducing the undirected versions of the three considered indices and give several equivalent formulations. We proceed by giving a short introduction of the most commonly used extensions of these indices to directed versions. We then present our own generalized versions, provide some technical results, and prove some properties of β-centrality. We end by illustrating some of the differences between the new indices and the traditional directed versions by applying them to known and artificial network datasets.

2 Preliminaries

In this work, we consider simple graphs $G = (V, E)$ with node set V and edge set E. A graph is directed if $E \subseteq \{(v, w) : v, w \in V\}$ and $(v, w) \in E$ does not imply $(w, v) \in E$. A graph is undirected if the edge set E consists of unordered pairs $\{v, w\}$. The cardinalities of V and E are denoted by n and m, respectively. While we focus on unweighted graphs, we note that most of our results admit a straightforward generalization to graphs with positive edge weights.

The neighborhood of a vertex $v \in V$ in an undirected graph is defined as $N(v) = \{w : \{v, w\} \in E\}$. For directed graphs, we define the out-neighborhood as $N^+(v) = \{w : (v, w) \in E\}$ and the in-neighborhood as $N^-(w) = \{w : (w, v) \in E\}$.

Every graph G can be represented by its adjacency matrix $A = (A_{ij})$, defined as

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{else.} \end{cases}$$

The matrix A is binary and symmetric if G is unweighted and undirected. In this case, all eigenvalues of A are real and we label the eigenvalues in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If G is connected, i.e. there exists a path between all pairs of vertices, then λ_1 is the dominant eigenvalue and $\lambda_1 > \lambda_2$ holds by the Perron-Frobenius theorem (Meyer, 2000). The eigendecomposition of symmetric and real-valued matrices A are given by $A = X\Lambda X^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $X = [\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}]$ is an orthogonal matrix, and $\boldsymbol{x_i}$ is the eigenvector associated with eigenvalue λ_i. The dominant eigenvector $\boldsymbol{x_1}$ can be chosen so that all entries are non-negative if G is symmetric and connected. The Perron-Frobenius theorem does not apply to general directed graphs and A may have non-real eigenvalues in this case.

A generalization of the eigendecomposition is given by the singular value decomposition (SVD). Note that it is generally defined for $n \times m$ matrices, but we apply it to the $n \times n$ adjacency matrix. The SVD is given by $A = U\Sigma V^T$, where $U = [\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}]$ are the left-singular vectors, $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is the matrix of singular values with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, and $V = [\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}]$ are the right-singular vectors.

Note that the right-singular vectors are eigenvectors of A^TA and the left singular-vectors of AA^T. The respective eigenvalues are given by the square roots of the singular values in Σ.

3 Undirected versions

We shall denote Hubbell[1] centrality by h, Katz centrality by k and Bonacich β-centrality by b. If A is an adjacency matrix of an undirected graph $G = (V, E)$ and $|\beta| < \lambda_1^{-1}$, then the measures can be expressed in matrix form as

$$\begin{aligned} \boldsymbol{h} &= (I - \beta A)^{-1}\mathbf{1} \\ \boldsymbol{k} &= \boldsymbol{h} - \mathbf{1} \\ \boldsymbol{b} &= (I - \beta A)^{-1}A\mathbf{1} \\ &= \frac{\boldsymbol{k}}{\beta} \quad \text{if } \beta \neq 0, \end{aligned} \tag{1}$$

where I is the identity matrix and $\mathbf{1}$ is a vector of all ones. Bonacich (2007) showed that there exists an important connection between β-centrality and eigenvector centrality. If A has a dominant eigenvalue λ_1 then, as $\beta \to |\lambda_1|^{-1}$ from below, $\boldsymbol{b} \to \boldsymbol{x_1}$. Thus, the rankings of eigenvector centrality and β-centrality will coincide. When $\beta = 0$, Hubbell's index is $\mathbf{1}$, Katz is $\mathbf{0}$ and β-centrality tends to degree centrality. Both Hubbell and Katz only considered positive values for β but Bonacich proposed to extend the range of β to negative values. While it is guaranteed that the entries of all three score vectors are non-negative for $0 < \beta < \lambda^{-1}$ since $(1 - \beta A)^{-1}$ is non-negative (Benzi and Klymko, 2015), this is not the case for negative values of β. That is, some entries may be negative while others are positive, making it hard to interpret the results. The usual guidance is to choose a value for β which produces only non-negative scores.

There exist several other ways of expressing the indices in Equation (1) which are important for the remainder of the paper. Given the relationship between the three indices, we only show alternatives for Hubbell. Others

[1]The version of Hubbell's index we are using is very simplistic but convenient for our purposes. The original definition is more general, including weights and exogenous effects.

then follow from the definitions in Equation (1). In the first formulation, we expand $(I - \beta A)^{-1}$ as a power series so that

$$\boldsymbol{h} = (I + \beta A + (\beta A)^2 + (\beta A)^3 + \ldots)\mathbf{1}. \tag{2}$$

This formulation highlights the connection of the indices with walk counts on the underlying graph. The term $\beta^k A^k_{ij}$ gives the number of walks between i and j of length k, attenuated by β^k. The smaller β, the less emphasis is put on longer walks.

Our second alternative is to express the centrality of an actor in purely graph theoretic terms. We can write Hubbell's index as

$$h(v) = 1 + \beta \sum_{w \in N(v)} h(w). \tag{3}$$

Equation (3) highlights the similarity in formulation of these measures and eigenvector centrality, in as much as the centrality of an actor is proportional to the centralities of the actors it is connected to.[2] A vertex thus needs to be connected to central vertices in order to be a central vertex itself. Indices of this kind are also referred to as feedback centralities.

To illustrate some practical aspects of the indices, specifically in terms of negative β's, we apply them to the well known business network of Florentine families (Padgett and Ansell, 1993). Figure 1a) shows the network of business relations among families and Figure 1b) the rankings for eigenvector centrality, and β-centrality with positive and negative β values.

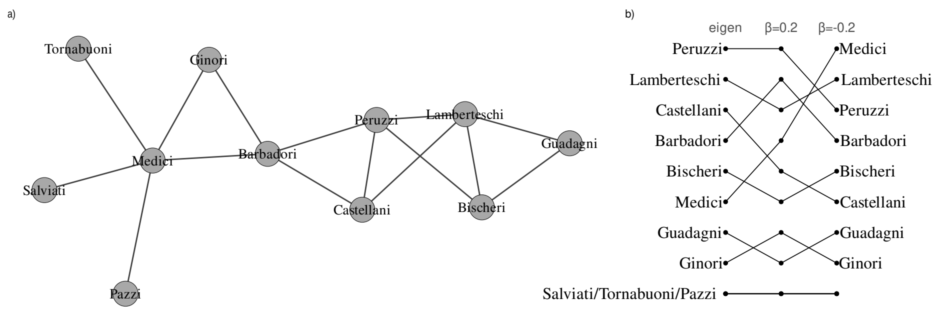

Figure 1: a) Florentine Family business network. Isolated families (Acciaiuoli, Albizzi, Pucci, Ridolfi, Strozzi) are not shown). b) Rankings of the Florentine families for eigenvector centrality, and β-centrality for $\beta = 0.2$ and $\beta = -0.2$ ($\lambda_1^{-1} = 0.31$).

The example highlights how the traditional interpretation of feedback centralities, i.e. being central means being connected to other central nodes,

[2]In the case of Hubbell, the values are proportional to the neighbors' centrality shifted by one.

may not adequately capture the strategic position of the Medici within this network. Eigenvector centrality as well as β-centrality with $\beta = 0.2$ indicate that the Medici family are not connected to other powerful families. Although they have the most business partners, these are less powerful than the business partners of the Barbadori and Peruzzi family. A negative β, however, reveals the power of the Medici in bargaining situations, where it is advantageous to be connected to others who have less bargaining possibilities (Cook et al., 1983). The Medici do rise to the top of the ranking for $\beta = -0.2$, suggesting that they have the highest bargaining power in the network. For three families, Tornabuoni, Salviati, and Pazzi, the Medici are the only business partners, giving them complete control over these families. The Peruzzi family drops to the third position, suggesting that their business partners are too powerful to be in an advantageous position.

4 Directed versions

It has been common practice to extend the introduced measures to directed data by replacing the symmetric matrix A with the asymmetric adjacency matrix corresponding to the directed graph. In addition, analogously to the simple in- and out-degree centrality indices, we can thus calculate in and out versions of $\boldsymbol{h}$, $\boldsymbol{k}$ and $\boldsymbol{b}$ by using A and A^T in Equations (1) and (2). We denote the in-versions by $\boldsymbol{h}^-$, $\boldsymbol{k}^-$, and $\boldsymbol{b}^-$ and the out-versions by $\boldsymbol{h}^+$, $\boldsymbol{k}^+$, and $\boldsymbol{b}^+$. The graph theoretic versions which extend Equation (3) for Hubbell's index to directed graphs are given by

$$\begin{aligned} h^-(v) &= 1 + \beta \sum_{w \in N^-(v)} h^-(w) \\ h^+(v) &= 1 + \beta \sum_{w \in N^+(v)} h^+(w), \end{aligned} \tag{4}$$

where $|\beta| < \lambda_1^{-1}$. The version of Katz and β-centrality can be obtained in a similar way.

This simple substitution of an asymmetric matrix and its transpose for the symmetric version of A has a number of consequences. First, in fixed or limited choice design, all actors have the same out-degree which results in the out-versions to give each actor the same score. Such data is quite common and occurs if egos are asked to select the same number of alters or if the network was obtained from dichotomizing rank ordered data with a fixed threshold (this is, e.g., often done with Newcomb's fraternity data (Newcomb, 1961)). Second, the in-score of an actor is proportional to the in-score of their in-alters and the out-score is proportional to the out-score of their out-alters. Technical results on the limiting behaviour of the indices for $\beta \to 0$ and $\beta \to \lambda^{-1}$ can be found in Benzi and Klymko (2015).

5 Defining in-out indices

Kleinberg's hubs and authorities (Kleinberg et al., 1999) yield an alternative approach to the indices derived in the previous section. These indices relate the in-score to the out-scores of their in-alters (authorities) and their out-score to the in-score of their out-alters (hubs), respectively. In graph-theoretic terms, similar to Equation (4), hubs and authorities can be defined as

$$\text{hub}(v) = \sum_{w \in N^+(v)} \text{auth}(w) \qquad \text{auth}(v) = \sum_{w \in N^-(v)} \text{hub}(w). \tag{5}$$

It can be shown that the so defined indices constitute an iterative power method to compute the dominant eigenvectors of $A^T A$ and AA^T (Benzi et al., 2013). Hence, authority scores are given by the dominant eigenvector of $A^T A$, that is the first right-singular vector $\boldsymbol{v_1}$ of A, and hub scores by the dominant eigenvector of AA^T, the first left-singular vector $\boldsymbol{u_1}$.

We can now adapt the indices defined in Equation (1) to capture the in/out relation as given by hubs and authorities. We call the corresponding measures i/o-Hubbell, i/o-Katz and i/o-β-centrality, where i/o represents in-out and we will denote them by h_i, h_o,k_i, k_o, b_i, b_o. We begin by defining i/o-Hubbell by extending Equation (3) and Equation (5) to

$$\begin{aligned} h_i(v) &= 1 + \beta \sum_{w \in N^-(v)} h_o(w) \\ h_o(v) &= 1 + \beta \sum_{w \in N^+(v)} h_i(w). \end{aligned} \tag{6}$$

Both in-out Katz and in-out β-centrality follow by applying Equation (1) to Equation (6). To get a high in-score for any of the in-measures with a positive β, an actor needs to be chosen by actors with high out-scores. To obtain a high out-score, again with positive β, an actor needs to choose actors with high in-scores. For negative β's the situation is reversed. To get a high in-score an actor needs to be chosen by actors with low out-scores and to get a high out-score an actor needs to choose actors with low in-scores. Note that it is possible to use different β with different signs in each of the equations but we do not explore this in any detail.

Equation (6) can be used to derive both a series version and a matrix version for Hubbell. The matrix versions are given by

$$\begin{aligned} \boldsymbol{h_i} &= (I - \beta^2 A^T A)^{-1}(I + \beta A^T)\mathbf{1} \\ \boldsymbol{h_o} &= (I - \beta^2 AA^T)^{-1}(I + \beta A)\mathbf{1}. \end{aligned} \tag{7}$$

This formulation is a proper generalization of $\boldsymbol{h}$ as given in Equation (1). If

A is symmetric then both right hand sides in Equation (7) reduce to

$$\begin{aligned}&(I-\beta^2A^2)^{-1}(I+\beta A)\mathbf{1}\\=&(I-\beta A)^{-1}(I+\beta A)^{-1}(I+\beta A)\mathbf{1}\\=&(I-\beta A)^{-1}\mathbf{1}=\boldsymbol{h}.\end{aligned}$$

The series formulation of Equation (7) can be derived by expanding the first term as a power series in the same way as in the undirected case.

Admissible choices of β can be obtained by similar considerations as in the undirected case. For $(I-\beta^2A^TA)^{-1}$ to be non-singular, we require β^2 to be smaller than the reciprocal of the modulus of the largest eigenvalue of AA^T. Equivalently, the reciprocal of β needs to be smaller in modulus that the largest singular value of A. That is, $|\beta|<\sigma_1^{-1}$.

It should be noted that Equation (7) is closely related to PN-centrality for signed networks proposed by Everett and Borgatti (2014). In their derivation $A=P-2N$, where P contains the positive and N the negative relations, and the value of β is fixed at $(2n-1)^{-1}$, where n is the number of vertices, in order to normalize the measure.

6 In-out β-centrality

Given the prominence of traditional β-centrality in the literature, we devote this section to a more rigorous mathematical treatment of its i/o-version. There are a few properties that carry over from the usual directed versions. The first is that as $\beta\to 0$, β-centrality tends to degree centrality and the usual directed versions to in- and out-degree, respectively. For the i/o-versions, it holds that

$$\begin{aligned}\boldsymbol{b_o}&=\frac{(\boldsymbol{h_o}-\mathbf{1})}{\beta}\\&=\frac{1}{\beta}(I-\beta^2AA^T)^{-1}(I+\beta A)\mathbf{1}-\frac{1}{\beta}\mathbf{1}\end{aligned}$$

and when $\beta\to 0$, the whole expression converges to $A\mathbf{1}$. Hence, $\boldsymbol{b_o}$ tends to out-degree. Analogously, $\boldsymbol{b_i}$ converges to in-degree.

Our second result is an extension of the fact that for a symmetric adjacency matrix A with dominant eigenvalue λ_1, then $\boldsymbol{b}\to\boldsymbol{x_1}$ for $\beta\to|\lambda_1|^{-1}$ from below. We show that for our extended directed i/o-indices, if A has a dominant singular value σ_1, then as $\beta\to\sigma_1^{-1}$ from below, then $\boldsymbol{b_i}$ and $\boldsymbol{b_o}$ tend to the left and right dominant singular vectors, denoted by u_1 and v_1, of A. This will show that the i/o-indices extend hubs and authorities in the same way as β-centrality extends eigenvector centrality for undirected networks.

Before we proceed, we note that there is no guarantee that a dominant singular value exists. Two examples of such graphs are shown in Figure 2.

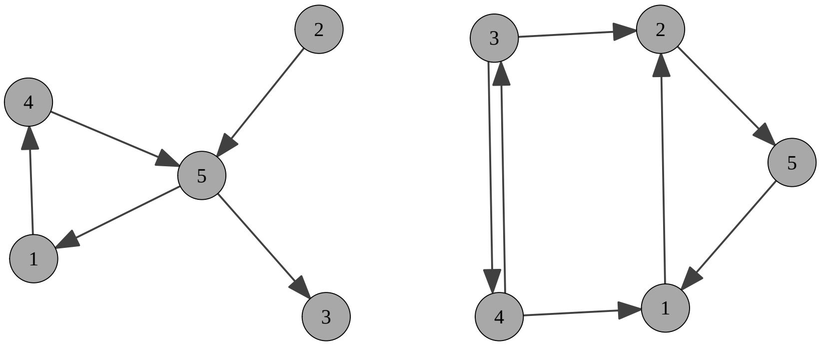

Figure 2: Two directed graphs without a dominant singular value. The singular values of the graph to the left are $[\sqrt{2}, \sqrt{2}, 1, 0, 0]$. Those of the graph on the right are $[\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, 1, \frac{2}{1+\sqrt{5}}, \frac{2}{1+\sqrt{5}}]$

We illustrate the convergence for $\boldsymbol{b_i}$ but note that the result for $\boldsymbol{b_o}$ can be derived in a similar manner. Let $A = U\Sigma V^T$ be the SVD of the adjacency matrix A associated with a directed graph G. We express A^T and $A^T A$ via the SVD of A as

$$A^T = \sum_{i=1}^{n} \sigma_i \boldsymbol{v_i} \boldsymbol{u_i}^T$$
$$A^T A = \sum_{i=1}^{n} \sigma_i^2 \boldsymbol{v_i} \boldsymbol{v_i}^T.$$

Recall that for $i \neq j$, $\boldsymbol{v_i}$ and $\boldsymbol{v_j}$ are orthonormal (as are $\boldsymbol{u_i}$ and $\boldsymbol{u_j}$). From this, it follows that

$$\begin{aligned}
\boldsymbol{b_i} &= \frac{1}{\beta}(I-\beta^2A^TA)^{-1}(I+\beta A^T)\mathbf{1} - \frac{1}{\beta}\mathbf{1} \\
&= \frac{1}{\beta}\sum_{k=0}^{\infty}(\beta^2A^TA)^k(I+\beta A^T)\mathbf{1} - \frac{1}{\beta}\mathbf{1} \\
\\
&= \frac{1}{\beta}\sum_{k=1}^{\infty}\beta^{2k}(A^TA)^k\mathbf{1} + \sum_{k=0}^{\infty}\beta^{2k}(A^TA)^kA^T\mathbf{1} \\
&= \frac{1}{\beta}\sum_{k=1}^{\infty}\beta^{2k}\sum_{i=1}^{n}\sigma_i^{2k}\boldsymbol{v_i}\boldsymbol{v_i}^T\mathbf{1} + \sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}\beta^{2k}\sigma_i^{2k}\boldsymbol{v_i}\boldsymbol{v_i}^T\right)\left(\sum_{i=1}^{n}\sigma_i\boldsymbol{v_i}\boldsymbol{u_i}^T\right)\mathbf{1}
\end{aligned}$$

.

Applying orthogonality to the second term, we obtain

$$\begin{aligned}
\ldots &= \frac{1}{\beta}\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty}(\beta^2\sigma_i^2)^k\right)\boldsymbol{v_i}\boldsymbol{v_i}^T\mathbf{1} + \sum_{i=1}^{n}\sum_{k=0}^{\infty}\beta^{2k}\sigma_i^{2k+1}\boldsymbol{v_i}\boldsymbol{v_i}^T\boldsymbol{v_i}\boldsymbol{u_i}^T\mathbf{1} \\
&= \sum_{i=1}^{n}\frac{\beta\sigma_i^2}{1-\beta^2\sigma_i^2}\boldsymbol{v_i}\boldsymbol{v_i}^T\mathbf{1} + \sum_{i=1}^{n}\frac{\sigma_i}{1-\beta^2\sigma_i^2}\boldsymbol{v_i}\boldsymbol{v_i}^T\boldsymbol{v_i}\boldsymbol{u_i}^T\mathbf{1}.
\end{aligned}$$

Note that $\boldsymbol{u_i}^T\mathbf{1}$, $\boldsymbol{v_i}^T\mathbf{1}$ are scalar and $\boldsymbol{v_i}^T\boldsymbol{v_i} = 1$ so that we finally obtain

$$\begin{aligned}
\ldots &= \sum_{i=1}^{n}\frac{(\boldsymbol{v_i}^T\mathbf{1})\beta\sigma_i^2}{1-\beta^2\sigma_i^2}\boldsymbol{v_i} + \sum_{i=1}^{n}\frac{(\boldsymbol{u_i}^T\mathbf{1})\sigma_i}{1-\beta^2\sigma_i^2}\boldsymbol{v_i} \\
&= \sum_{i=1}^{n}\frac{\sigma_i\left(\boldsymbol{v_i}^T\mathbf{1}\beta\sigma_i + \boldsymbol{u_i}^T\mathbf{1}\right)}{1-\beta^2\sigma_i^2}\boldsymbol{v_i}
\end{aligned}$$

If A has a dominant singular value σ_1, then we can choose $\boldsymbol{v_1}$, and also $\boldsymbol{u_1}$, so that all entries are non-negative. As β is positive, the above expression will tend to $\boldsymbol{v_1}$ as $\beta \to \sigma_1^{-1}$. That is, the ranking converges to the authorities ranking. The situation for $\beta \to -\sigma_1^{-1}$ is more involved. Firstly, if $\boldsymbol{u_1}^T\mathbf{1} = \boldsymbol{v_1}^T\mathbf{1}$, then the first term (i.e. $i = 1$) will vanish and so we will not converge to $\boldsymbol{v_1}$. If $\boldsymbol{u_1}^T\mathbf{1} \neq \boldsymbol{v_1}^T\mathbf{1}$, then $\boldsymbol{b_i}$ will converge to either $\boldsymbol{v_1}$ or $-\boldsymbol{v_1}$, depending on whether $\boldsymbol{u_1}^T\mathbf{1}$ is greater or less than $\boldsymbol{v_1}^T\mathbf{1}$. It follows that if $\boldsymbol{b_i}$ is positive, then $\boldsymbol{b_o}$ will be negative and vice versa. We have, however, implicitly assumed that our centrality scores are non-negative and so we would not consider values of β that give negative values to be valid. We also note that convergence can be very slow if $\boldsymbol{u_1}^T\mathbf{1}$ is nearly equal to $\boldsymbol{v_1}^T\mathbf{1}$.

We know that as $\beta \to -\sigma_1^{-1}$, we will eventually have negative values in either $\boldsymbol{b_i}$ or $\boldsymbol{b_o}$. As $\beta = 0$ gives in- and out-degree, there must be a value of β, i.e. β^* , such that if $\beta < \beta^*$ then either $\boldsymbol{b_i}$ or $\boldsymbol{b_o}$ have negative values but if $\beta \geq \beta^*$ then $\boldsymbol{b_i}$ and $\boldsymbol{b_o}$ are non-negative. To our knowledge there is no result which helps us determine β^* other than trial and error. As a consequence we strongly advise that the selection of any negative β should be driven by obtaining non-negative entries for $\boldsymbol{b_i}$ and $\boldsymbol{b_o}$.

7 Examples

In this section, we highlight some of the differences and peculiarities of the newly defined indices by means of two well known datasets and constructed graphs.

To highlight some of the issues of the directed version of traditional β-centrality, we look at the campnet data which is available in UCINET (Borgatti et al., 2002). The network is shown in Figure 3. Every actor has

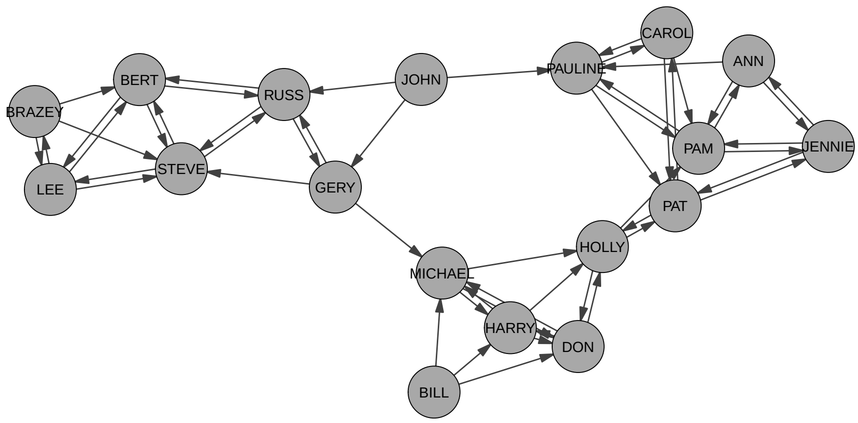

Figure 3: Campnet nomination network.

selected three alters in this dataset such that all out-degrees are equal to three. The dominant singular value of A is 3.376 ($\sigma_1^{-1} \approx 0.296$). We applied both the traditional directed versions of β-centrality and i/o-Hubbell with $\beta = \pm 0.2$. This particular value of β was chosen since all scores remain positive and it is far enough from the lower bound to ensure no convergence issues. No substantive meaning should be given to this particular parameter choice.

As predicted, b^+ does not give any useful information, neither for positive nor negative β. In contrast, our new measure h_o is able to distinguish the nodes, regardless of the fact that they all have the same out-degree. This ranking, however, is the same for positive and negative β. The rankings induced by b^- for positive and negative β only vary moderately (Kendall's $\tau = 0.65$). Interestingly, the ranking induced by b_i for positive β is more

Table 1: different directed centrality scores for the campnet data.

	b^-		b^+		h_i		h_o	
	$\beta = 0.2$	$\beta = -0.2$	$\beta = 0.2$	$\beta = -0.2$	$\beta = 0.2$	$\beta = -0.2$	$\beta = 0.2$	$\beta = -0.2$
HOLLY	10.857	2.031	7.5	1.86	3.275	0.431	3.170	0.793
BRAZEY	2.586	0.650	7.5	1.86	1.553	0.862	3.012	0.753
CAROL	6.047	0.910	7.5	1.86	2.103	0.724	3.152	0.788
PAM	13.356	3.289	7.5	1.86	4.028	0.243	2.610	0.652
PAT	11.054	2.492	7.5	1.86	3.429	0.393	2.604	0.651
JENNIE	9.183	1.641	7.5	1.86	2.642	0.590	2.912	0.728
PAULINE	9.182	2.957	7.5	1.86	3.303	0.424	2.912	0.728
ANN	6.508	1.014	7.5	1.86	2.104	0.724	2.995	0.749
MICHAEL	7.844	2.790	7.5	1.86	3.387	0.403	2.880	0.720
BILL	0.000	0.000	7.5	1.86	1.000	1.000	2.902	0.726
LEE	7.928	1.752	7.5	1.86	2.785	0.554	2.766	0.691
DON	9.015	2.654	7.5	1.86	3.393	0.402	2.879	0.720
JOHN	0.000	0.000	7.5	1.86	1.000	1.000	2.759	0.690
HARRY	6.372	1.911	7.5	1.86	2.732	0.567	3.011	0.753
GERY	3.836	1.483	7.5	1.86	2.128	0.718	3.143	0.786
STEVE	11.764	3.235	7.5	1.86	3.965	0.259	2.892	0.723
BERT	10.291	2.356	7.5	1.86	3.310	0.423	3.023	0.756
RUSS	9.178	2.585	7.5	1.86	3.363	0.409	2.881	0.720

correlated with the ranking of b^- for negative β ($\tau = 0.87$) than for positive β ($\tau = 0.70$). Additionally, the ranking of b_i for $\beta = -0.2$ is a perfect reversal of the ranking for $\beta = 0.2$.

Next, we consider a constructed class of graphs which illustrates that the i/o-indices can distinguish vertices even when both degree and hubs and authorities are (almost) constant. The class consists of graphs with $m > 1$ components, where each component is a complete bipartite graph with $n = 1, 2, \dots, m$ vertices in the left and two vertices in the right partition. All edges in each component point from the n left nodes to the two right nodes. It is clear that all vertices in left partitions have out-degree two and all nodes in right partitions have out-degree zero for any $m > 1$. For hubs, we have zero for all nodes except in the largest component where the left partition has a score of one. Hence, the discriminant power of out-degree and hubs is very low for this class of graphs. Any of the i/o indices, on the other hand, does discriminate between the left-hand vertices in different partition, ranking them non-decreasingly according to n.

As a second empirical example, we turn to the advice network of high-tech managers collected by Krackhardt (1987). A directed tie between manager v and w indicates that manager v seeks advice from w (cf. Figure 4 top). The network is drawn such that the x-coordinates correspond to in-degree and the y-coordinates to out-degree. Hence, manager 2 is being asked for advice the most and manager 15 seeks the most advice. Employing an i/o-index, in this case again the version of Hubbell, can be interpreted as follows. The value $h_i(v)$ is high for $\beta > 0$, if the manager v gives advice to managers that seek a lot of advice in general. In the case of $\beta < 0$, $h_i(v)$ is high if they give advice to others who do not seek much advice overall. The score $h_o(v)$ is high for $\beta > 0$, if the manager v seeks a lot of advice from managers that give a lot of advice in general. For $\beta < 0$, the score is high if the manager seeks advice from others who do not give out much advice.

The bottom of Figure 4 shows the induced rankings by b^- and b^+ for both a positive and negative value of β, and the rankings induced by h_i and h_o, again for positive and negative values of β. The β values were again chosen for purely technical reasons, i.e. to ensure that all scores for all indicies are positive and bear no substantive meaning.

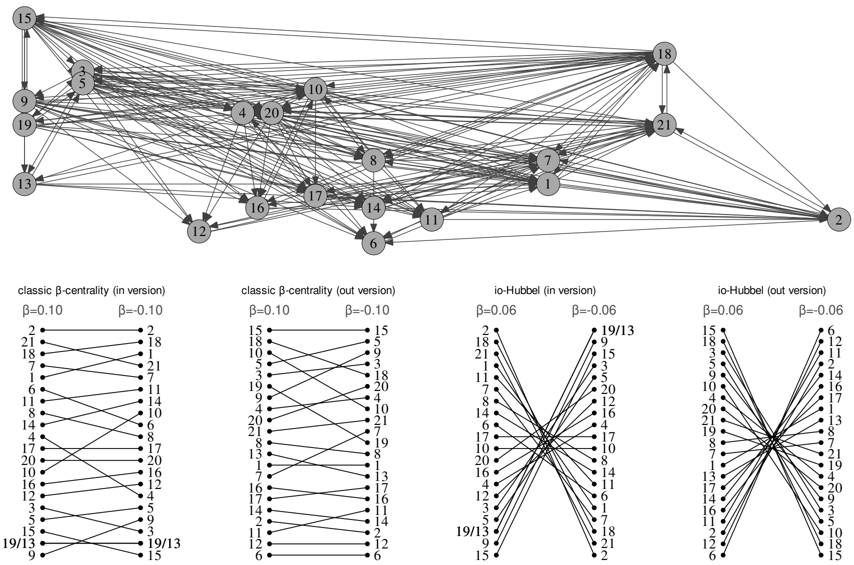

Figure 4: Top shows Krackhardt's high-tech managers advice network ("who do you go to for advice?"). x-cordinates are proportional to in-degree and y-coordinates to out-degree. Bottom shows the rankings for traditional β-centrality and the i/o-Hubbell index (h_i and h_o) for positive and negative β values ($|\lambda_1|^{-1} \approx 0.121$ and $\sigma_1^{-1} \approx 0.09$).

We can observe that for positive β, the rankings of b^- and h_i, as well as b^+ and h_o are fairly comparable. This is not too surprising, since they have similar convergence patterns. However, looking at negative β, the commonalities disappear. While the rankings of traditional β-centrality do not change significantly, there are drastic changes for the i/o-index. The rankings do not reverse entirely, but many managers that are ranked at the bottom for b^- with $\beta < 0$ are ranked at the top of h_i. Similar for the out versions of the respective indices.

The examples illustrate that the proposed i/o-indices can pick up different signals than their traditional counterparts. Recall, though, that for $\beta \to 0$, b_i will converge to in-degree and for $\beta \to \sigma_1^{-1}$ to the authorities ranking. Analogously, b_o will tend to out-degree and hubs. We thus expect high correlations with the indices at the respective limits. To investigate the behavior for intermediate parameter settings, we use 100 equidistant and increasing values for β from the interval $(0, 0.99\sigma_1^{-1}]$, for both the campnet and high-tech managers data.

In the campnet network, the rank correlation between authorities and b_i never drops below 0.967 and for in-degree never below 0.903. Similar, the correlation between hubs and b_o is always higher than 0.804. The out-degree is constant, such that the correlation is undefined. For the high-tech managers, the minimum rank correlations are 0.952 and 0.914 for authorities and in-degree with b_i, and 0.943 and 0.935 for hubs and out-degree with b_o. Thus, there is very little variation between the limiting cases. We note that this is only an artefact of the data and not a general property of the indices. Figure 5 shows a graph with 100 nodes, sampled from the Barabási-Albert model (Barabási and Albert, 1999) where each edge was randomly rewired with a probability of 0.2. The correlations between b_i and b_o and their respective limiting indices, shown in the right panel, indicate that it is indeed possible to obtain lower correlations for $\beta \in (0, \sigma_1^{-1})$ yielding new opportunities to pick up different signals than in- and out-degree or hubs and authorities do.

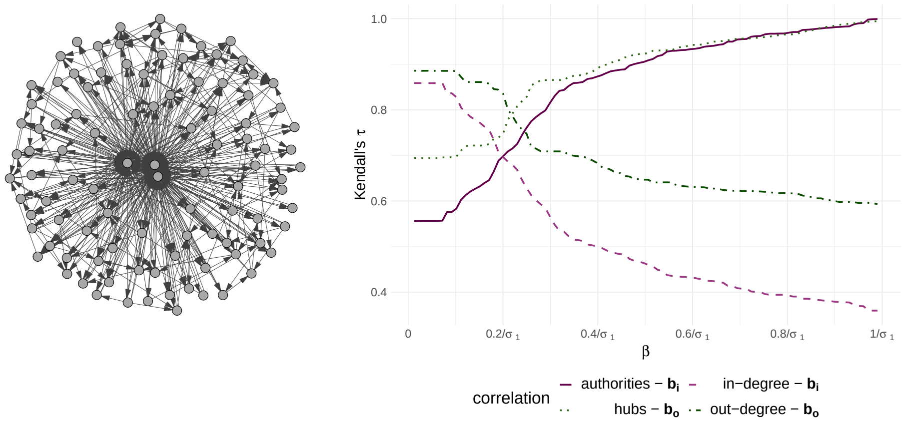

Figure 5: Randomly rewired network derived from the Barabási-Albert model and the rank correlation of i/o-β-centrality with limiting cases for $\beta \to 0$ and $\beta \to \sigma_1^{-1}$.

8 Conclusion

We extended the conceptual idea of β-centrality in directed networks by relating the in-centrality of ego to the out-centrality of their in-neighbors and the out-centrality to the in-centrality of out-neighbors. The so defined indices thus complete the family of measures proposed by Bonacich (1987), where in-centrality is related to the in-centrality of in-neighbors and out-centrality is related to the out-centrality of out-neighbors. Each of the indices comes with a tunable parameter β which can either be positive or negative. While positive values for β follow the usual interpretation of feedback centrality indices, negative values indicate power in, e.g., exchange networks as described by Cook et al. (1983). Our examples illustrate that the newly defined indices can pick up different signals than the traditional directed versions of β-centrality. In the case of constant in- or out-degrees, the respective directed index is unable to distinguish the nodes. Our i/o-indices, on the other hand, are able to rank nodes even if one set of degrees is constant. Additionally, the examples show that the rankings for positive and negative choices of β can vary significantly.

The mathematical treatment of the indices show that for positive β's, the rankings for the in- and out-versions will converge to the ranking of the principal singular vectors, that is to hubs and authorities, if $\beta \to \sigma^{-1}$. This generalizes the known result for the undirected case, where the rankings converge to the principal eigenvector. Hence, the i/o-indices generalize hubs and authorities in the same way as β-centrality generalizes eigenvector centrality.

The situation for negative β's is slightly more involved and the convergence depends on the magnitudes of the principal singular vectors. Additional experiments with random graphs suggest that the rankings become very instable if β is chosen to be very close to $-\sigma_1^{-1}$. It is thus recommended to use values for β which are not too close to the extreme end. This also ensures that all scores remain positive. We deem the following three strategies to find acceptable parameter settings as viable options in the absence of substantive arguments. First, pick a β that produces all positive scores for $-\beta$ as was done in our illustration. Second, pick a value halfway between 0 and σ_1^{-1} as that has a good chance of producing different results from degree and hubs and authorities. Third, it is possible to run a range of parameters and pick the values that correlate least with degree and hubs and authorities. It must be emphasized, that no formal result exists for the choice of β which guarantees the positivity of scores. In the absence of such a proof, it is necessary to experiment with different negative β values in empirical settings which guarantee non-negative entries for all score vectors using, for instance, the outlined strategies.

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